Fermi-Dirac statistics and traffic in complex networks

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(Received 11 November 2004; revised manuscript received 25 January 2005; published 16 June 2005)

We propose an idealized model for traffic in a network, in which many particles move randomly from node to node, following the network's links, and it is assumed that at most one particle can occupy any given node. This is intended to mimic the finite forwarding capacity of nodes in communication networks, thereby allowing the possibility of congestion and jamming phenomena. We show that the particles behave like free fermions, with appropriately defined energy-level structure and temperature. The statistical properties of this system are thus given by the corresponding Fermi-Dirac distribution. We use this to obtain analytical expressions for dynamical quantities of interest, such as the mean occupation of each node and the transport efficiency, for different network topologies and particle densities. We show that the subnetwork of free nodes always fragments into small isolated clusters for a sufficiently large number of particles, implying a communication breakdown at some density for all network topologies. These results are compared to direct simulations.

DOI: 10.1103/PhysRevE.71.066114 PACS number(s): 89.75.Fb

I. INTRODUCTION

The subject of complex networks has become an important area of research in statistical physics since its debut with the paper by Watts and Strogatz [1] (for reviews, see Refs. [2–4]). As a result of past work in the area, we now know that most of the networks, either natural or manmade, are *scale-free* [5], with a very inhomogeneous distribution of the number of neighbors among nodes, following a power law. A natural question is: how do networks' structural features affect the dynamics of processes taking place within the network? The study of the dynamics of complex networks is seen nowadays as one of the main challenges in this area.

One of the most natural and important kind of dynamical process to consider is the flow of traffic through a network. A typical example is the flow of information in a communication network, such as the Internet. In the Internet, any message is first split into small packets, each of which follows independently its own trajectory, hopping from node to node until they find their destiny. One can consider this process as a number of particles moving in the network through its links. As a first approach, one can imagine that the particles do not interact with each other, or equivalently, one can consider the motion of a single particle in the network. This situation has been studied in a number of previous works. The particle can be thought of as searching through the network for its target node, using only local information [6–13] (it is worth mentioning that in Ref. [9] a search method is adapted to take particle interaction into account). Another approach is to consider that the particle is performing a random walk in the network, and study the properties of the resulting diffusion process [14–20].

These "noninteracting" approaches yield valuable insight on the full problem of traffic, besides being important in their own right. However, in order to fully understand the dynamics of network traffic, sooner or later the interaction between particles has to be taken into account. In particular, the nodes of real networks do not have infinite processing capacity, and if there are many particles (Internet "packets," for example), they will get into each other's way, causing collective behavior such as congestion and jamming. One of the ways to take this into account is by measuring the importance of each node to transport, motivating the introduction of *centrality measures*, of which the most prominent is the *betweenness centrality* [21–26]. A different approach is the direct simulation of traffic, usually by cellular-automata-like models [24–35]. Recent models have also considered cascades of malfunctions caused by congestion [36].

We feel that it would be highly desirable to have a simple model of transport which incorporates collective effects like congestion in a tractable way, such that some analytic results could be obtained for the statistics of the system. Even though such model would necessarily have to be highly idealized, it would serve as a tool to explore the dynamics of transport in real networks. This follows the tradition of statistical physics, which has greatly benefitted from the use of idealized models, such as the Ising model for magnetic systems, and many others. Motivated by this, we introduce in this paper a simple model of traffic in a network, which incorporates the collective effects in the simplest possible way. In our model, there are n particles moving randomly in the network, subject to the constraint that each node can be occupied by at most one particle. Other than this constraint, each particle performs a random walk (details of the model are given in Sec. II). This avoidance rule is inspired by the limited capacity of nodes in communication networks to process information. In Sec. III, we show that this constraint works like Pauli's exclusion principle of quantum mechanics. As a consequence, the particles behave like a set of n free fermions on a quantum system with N single-particle states (where N is the number of nodes of the network), with appropriately chosen energy levels. The statistical properties of the model are thus given by the Fermi-Dirac distribution of the equivalent quantum system. As a result, analytical expressions are found for all the statistical quantities of interest. To the best of our knowledge, this is the first model of

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transport in networks with inter-packet interaction which yields analytical results. Although it is certainly very idealized, it incorporates collective effects such as jamming in a natural way. The appearance of a Fermi-Dirac statistics in this model establishes a nice connection between transport problems in networks and equilibrium quantum statistical mechanics, which perhaps can be generalized to more realistic transport models. As a testing ground for our theory, a direct simulation of the traffic process described above is performed in a network grown according to the Barabási-Albert scheme, and the numerical results obtained thereby are found to be in excellent agreement with those predicted by the Fermi-Dirac distribution.

We want to use this theory to get information on traffic flow, especially collective effects such as jamming. With this motivation in mind, we define in Sec. IV a quantity Q proportional to the number of failed hoppings due to the occupation of the nodes, and find an analytical expression for it using the Fermi-Dirac statistics. We find that the inhomogeneity of the degree distribution tends to decrease the transport efficiency of the network. This can be understood from the fact that the most connected nodes are the ones most likely to be occupied, and they are exactly the most crucial nodes to transport.

The structure of the subnetwork of occupied nodes is investigated in Sec. V. The condition for the percolation of this subnetwork is found in terms of the density of particles (number of particles divided by the number of nodes of the network). We apply this condition to the particular case of a scale-free network, with a degree distribution given by a power law, $p_k \sim k^{-\gamma}$. We find that for $\gamma \leqslant 3$, the occupied nodes always percolate, even for arbitrarily small densities (in the limit of large network size). Thus, in these networks, the particles always aggregate in a giant cluster consisting of (on average) highly connected occupied nodes.

Another important issue is the structure of the free (unoccupied) nodes, also discussed in Sec. V. We apply the Fermi-Dirac formalism to find the percolation condition of the free subnetwork. The result is that there is always a percolating transition, for any network topology: above a certain critical density of particles (which depends on the degree distribution), the free subnetwork no longer percolates. This means that for high densities, the free nodes are scattered among small fragmented noncommunicating clusters. The free subnetwork is the set of nodes in a communication network through which a new "packet" can pass unhindered by the congestion. Thus, for densities above this percolating transition, there is a global breakdown in communication.

The final section of the paper, Sec. VI, summarizes our results, with a discussion on their significance and their possible generalization.

II. MODEL

In this paper we consider a network with N nodes. We assume the network to be connected, that is, there is a path, built with network links, connecting any two nodes. The network is defined by the probabilities p_k that a randomly chosen node has k neighbors (i.e., that the node has degree k). Our theory is valid for any distribution p_k, so we do not make any assumption about it at this point.

We assume that information flows in the network in discrete packets. We represent this by $n < N$ "packets" or "particles" moving in the network. These particles can be thought of as representing data packets in the Internet, for example. We further assume that each node can be occupied by at most one particle. This is the simplest way to incorporate into the model a finite capacity of each node. This prescription amounts to an effective inter-particle interaction: particles "avoid" each other.

In order to completely define the model, we must prescribe how the particles move within the network. We define a very simple stochastic dynamics. At each time step, one of the n particles is chosen randomly, with uniform probability among all particles. The chosen particle is at this moment in some node, which is linked to k other nodes (its neighbors). The particle chooses one of these k neighbors randomly, and tries to move there. If the chosen node is unoccupied, the move is successful, and the node becomes occupied by the particle, which leaves its previous node free. If, however, the target node is already occupied, the move fails, and the particle remains where it is. After this, a new time step begins, a new particle is selected randomly, and so on.

In the case when there is only a single particle in the network ($n=1$), the above dynamics means that this particle executes a random walk in the network. If $n>1$, however, the particles interfere with each other's motion. This interference is a direct result of our restriction that there can be only one particle in a node at any given time. There results from this a kind of "interaction" between the particles, which is responsible for the appearance of collective phenomena such as jamming. The model presented here introduces this interaction in the simplest possible way, which allows us to understand in detail the system's behavior, and to get some important analytical results, as we shall see in the following.

III. FERMI-DIRAC STATISTICS OF THE TRAFFIC DYNAMICS

As mentioned in the previous section, a single particle following the dynamics of our model performs a random walk in the network. It is a well-known result from graph theory that in this case, in the limit of long times, the time the particle spends in a given node is proportional to the degree k of the node [37]. In other words, the probability q_i that the particle is found in a node i of degree k_i is proportional to k_i:

$$q_i = \frac{k_i}{\sum_j k_j}, \tag{1}$$

where the sum is over all nodes of the network.

This gives us the occupation probability of any node in the case of one particle, $n=1$. What happens if there are many particles in the network? The particles interact with each other only through the restriction that the occupation of any node is at most 1. Exactly the same situation occurs in a system with n fermions, consisting of N single-particle

states. No two fermions are allowed to occupy the same quantum state. If it is assumed that the n fermions do not interact directly (i.e., they are free), this restriction gives rise to Fermi-Dirac distribution [38]. Our particles are an exactly analogous system: the particles (or "packets") are the equivalent of fermions, and the nodes are analogous to quantum single-particle states. This analogy leads us to conclude that *the occupation of the nodes (after equilibrium is reached) is given by the Fermi-Dirac distribution*.

In order to derive results from this analogy, we need to know the equivalent of the "energy" associated with each node, that is, the energy of the single-particle states. This is done by using the result from statistical physics that the one-particle occupation probability of a state i is proportional to the Boltzmann factor:

$$q_i = C_0 e^{-\beta E_i}, \tag{2}$$

where E_i is the energy associated with the state i, β is the inverse temperature, and C_0 is a normalization constant. The single-particle occupation probability in our case corresponds to one particle moving through the network, and is given by Eq. (1). Substituting this in the above expression, we obtain the expression for the "energy" associated with a node i with degree k_i:

$$\beta E_i = \ln(C k_i^{-1}). \tag{3}$$

Here C is a constant. Notice that Eq. (3) determines only the product βE_i, not β or E_i separately. This means that in our model the "temperature" $1/\beta$ is not fixed, and cannot be thought of as an independent external parameter, as is usually the case in statistical physics.

We denote by n_i the mean occupation of node i, when there are n particles in the network. As usual in the physics of many-particle systems, we use the grand canonical ensemble, where the number of particles is not fixed, but its mean value is prescribed. n_i is then given by the Fermi-Dirac distribution:

$$n_i = \frac{1}{e^{\alpha+\beta E_i}+1}, \tag{4}$$

where βE_i is given by Eq. (3) and α is the chemical potential. Substituting Eq. (3), we have

$$n_i = \frac{1}{A k_i^{-1}+1}, \tag{5}$$

where A is related to the chemical potential, and is determined by requiring that the average number of particles in the network be equal to the prescribed value n:

$$\sum_i \frac{1}{A k_i^{-1}+1} = n. \tag{6}$$

For a network with degree distribution p_k, the number of nodes with degree k is Np_k. Denoting by $\mu = n/N$ the occupation density, we rewrite the condition determining A in terms of a sum over degrees (instead of over nodes):

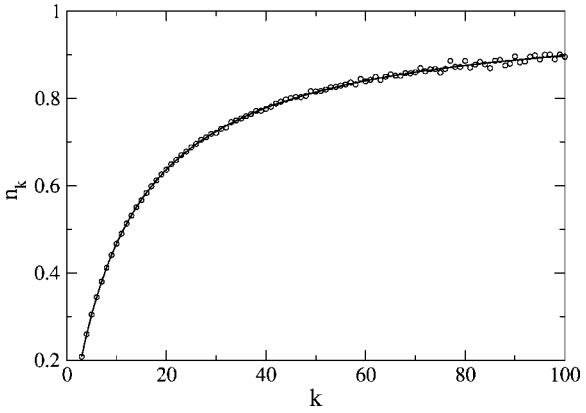

FIG. 1. Mean occupation n_k of nodes with degree k, in a Barabási-Álbert network with 10^5 nodes, and 3×10^4 particles, corresponding to $A\approx 11.42$. Circles are values measured from the simulation, and the continuous line is the prediction of Eq. (6).

$$\sum_k \frac{p_k}{A(\mu)k^{-1}+1} = \mu, \tag{7}$$

where we indicate explicitly that A depends on μ. From the above equation, we can show that $A(\mu)$ is a positive decreasing function of μ, $dA/d\mu<0$. Using the fact that the p_k distribution is normalized ($\Sigma_k p_k=1$), and Eq. (7), we can further show that

$$\lim_{\mu\to 0} A(\mu) = \infty; \quad \lim_{\mu\to 1} A(\mu) = 0. \tag{8}$$

We note that even though the number of particles n (and therefore μ) is not exactly constant in the grand canonical ensemble, for large n its fluctuations are negligible. Thus, in the thermodynamic limit ($n, N\to\infty$, $n/N\to\mu$), the results are the same as those obtained using the canonical ensemble (which fixes n).

From Eq. (5), we see that the mean occupation of a node increases with its degree, as was to be expected. Moreover, as $k\to\infty$, $n_k\to 1$: if a node has many neighbors, it is almost certain that it will be occupied by a particle. This "clustering" of particles in the more connected nodes leads to jamming, as we shall see.

In order to test Eq. (5), we simulate the stochastic dynamics described in Sec. II. We generate a scale-free network by the Barabási-Albert algorithm [5], with $N=10^5$ nodes, with an average of 6 links per node. We release $n=3\times10^4$ particles in it, initially randomly distributed among all nodes. This corresponds to $\mu=0.3$. After a sufficiently long time (enough to reach dynamical equilibrium), we measure the mean occupation of the nodes as a function of their degree k. The result is shown in Fig. 1 as the circles. To compare the results of the simulation with the theoretical prediction of Eq. (5), we determine the value of the constant A by numerically solving Eq. (7), finding $A\approx 11.42$ for $\mu=0.3$. Substituting this value in Eq. (5), we plot the distribution n_k of mean occupation for nodes with degree k. This function is plotted in Fig. 1 as the thick line. We see that the theory matches the simulated results almost perfectly, confirming that the Fermi-

Dirac distribution indeed describes the statistics of the particles.

IV. JAMMING

In the particle dynamics defined in Sec. II, if a particle attempts to move to an occupied node, the move fails and the particle stays where it is, no motion occurring in that time step. Let us call P the probability that this happens (after transients are through and dynamical equilibrium is reached), given by the ratio of the number of time steps where this takes place to the total number of time steps, in a long enough run. P provides us with a quantitative measure of the amount of jamming, and we investigate its behavior in this Section.

In this and the following section, we assume that the network structure is that of a *generalized random network*, in which the links are taken to be randomly distributed among the nodes, subject to the constraint that the degree distribution p_k be equal to the prescribed distribution [39]. Even though most existent networks have correlations not present in the equivalent random network, it is found that in most cases they are well described by assuming they are random, with only small discrepancies. In the particular case of the Barabási-Albert network, which we again use as an example, small correlations are known to be present [40], but we neglect them here. As we shall see, this gives a good approximation for the transport dynamics.

To calculate P, we first define the probability Q_k that, following a randomly chosen link, we arrive at a node with degree k. For a network described by a degree distribution p_k, in the random approximation discussed above, it is given by [39]

$$Q_k = \frac{kp_k}{\sum_m mp_m}. \tag{9}$$

In each time step, we choose successively a particle and a neighbor of this particle. If we assume that the distribution of links is not correlated to the degree distribution of the corresponding nodes (that is, we do not consider assortativity or dessortativity [41]), this amounts to choosing randomly a link in the network. Thus, the probability P that we arrive at an occupied node is

$$P = \sum_k Q_k n_k = \frac{1}{\langle k \rangle} \sum_k \frac{kp_k}{Ak^{-1} + 1}, \tag{10}$$

where $\langle k \rangle = \Sigma_k kp_k$ is the mean degree of the network.

We check the prediction of Eq. (10) by a simulation, again on a Barabási-Albert network. By means of Eqs. (7) and (10), we can calculate the prediction of our theory for P as a function of the particle density μ. For each value of μ we have to solve Eq. (7) for A, and then do the sum in Eq. (10) numerically. The result is shown in Fig. 2 (the continuous line). The circles in that figure are the values of $P(\mu)$ computed from the simulation. Again, the agreement between theory and simulation is very good, although Eq. (10) seems to be consistently a little below the observed results. This may be due to correlations arising from the growth process, which are neglected in expression (10).

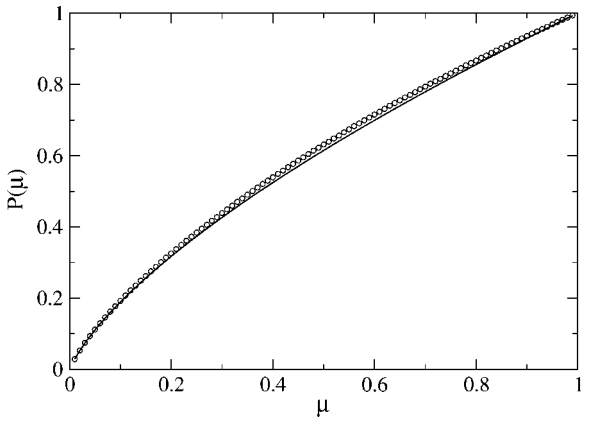

FIG. 2. Jamming probability P as a function of the particle density μ, for a Barabási-Albert network with $N=10^4$ nodes. Circles are values measured from the simulation, and the line is the prediction of Eq. (10).

Notice that $P(\mu) > \mu$ throughout the plot. If the particles occupied the nodes with uniform probability, we would expect $P(\mu) = \mu$. However, from Eq. (5), the nodes with greater degree are more likely to be occupied. This implies that a disproportionate number of links lead to occupied nodes (on average). This is the reason why $P(\mu) > \mu$. This phenomenon happens for any network with an inhomogeneous degree distribution, and should be more pronounced the greater the departure from homogeneity is. However, we must always have $P \to 1$ as $\mu \to 1$.

V. PERCOLATION

At any given time, the nodes of the network can be characterized as occupied or free. We thus have two well-defined subnetworks: one composed of occupied nodes, and one defined by unoccupied ones. An important question is whether these subnetworks percolate or not. In particular, if the subnetwork of free nodes percolates, it means that a finite fraction of the remaining unoccupied nodes forms a single connected giant cluster. We can interpret this as a situation in which "communication" is possible though a finite fraction of the network. If, on the other hand, the free subnetwork does not percolate, the unoccupied nodes form a fragmented mutually disconnected collection of small clusters, each of which having a vanishing fraction of the total number of nodes. In this case, global communication breaks down in the network. We expect that, as the particle density μ increases, this transition from percolated to a nonpercolated state will occur. Of particular importance is the critical value μ_c of the density, in which the transition occurs.

According to our model, the free and occupied subnetworks are constantly changing in time, even after dynamical equilibrium is reached. However, we expect that in the thermodynamic limit (large networks), the existence (or otherwise) of percolation in these subnetworks does not depend on time (although the precise structure of the percolated cluster does depend on time).

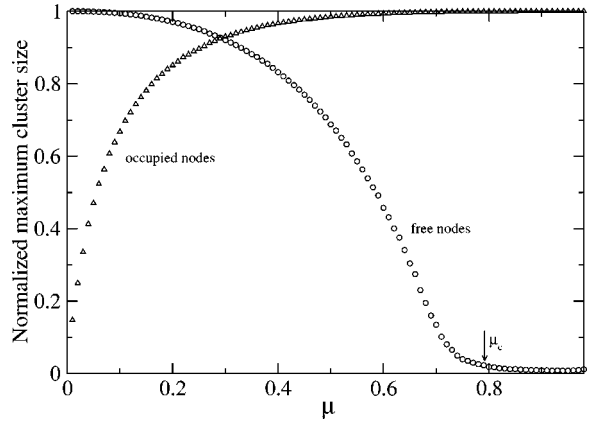

FIG. 3. Normalized size of the largest cluster S as a function of the particle density μ, averaged over 100 realizations, for the unoccupied (free) subnetwork (circles), and for the occupied subnetwork (triangles). The simulation was done on a Barabási-Albert network, with $N=10^4$ nodes. The arrow shows the theoretical prediction for the critical density μ_c.

Let ρ_k be the probability that a node with degree k is "active" in some sense. For example, "active" might mean occupied, or free. For generalized random networks, the subnetwork defined by the "active" nodes percolates if $\Sigma_k k(k-1)\rho_k p_k \geq \langle k\rangle = \Sigma_k k p_k$ [39]. The critical state, when the system just percolates, is thus given by

$$\sum_k \{k(k-1)\rho_k - k\}p_k = 0, \tag{11}$$

which determines the percolation transition. We are mostly interested in the percolation properties of the unoccupied (free) subnetwork, for which $\rho_k = 1 - n_k$. Using Eq. (5), we find the condition for the transition:

$$\sum_k \left\{ \frac{k(k-1)}{1 + k/A(\mu_c)} - k \right\} p_k = 0. \tag{12}$$

This equation, together with Eq. (7), must be solved for the critical particle density μ_c, for a given degree distribution. We notice that, since $A(\mu)$ is strictly decreasing (see Sec. III), there is at most one solution.

We again test these results by means of a numerically generated Barabási-Albert network, with the same parameters cited in Sec. III. Using the p_k from the generated network, we solve Eq. (12) numerically. We find the value $\mu_c \approx 0.79$. To check this, we simulate the stochastic dynamics of the particles (see Sec. II) for different values of μ, and record the normalized size S of the largest cluster of the free subnetwork (that is, number of nodes in the largest cluster divided by the total number of nodes in the subnetwork), averaged over a number of realizations (100, in this example). We thus obtain a function $S(\mu)$. In the percolated state, $S>0$, as a finite fraction of the nodes belong to the largest cluster. In the non-percolated state, $S=0$. Hence, we expect $S(\mu)$ to be flat and equal to zero for $\mu > \mu_c \approx 0.79$, and suddenly rise to nonzero values for $\mu < \mu_c$. As can be seen in Fig. 3, this is indeed what happens. In particular, notice that the critical density μ_c predicted by the theory agrees very well with the results from the simulation.

We can show that the free subnetwork always has a critical density μ_c between 0 and 1, for any connected network with average degree greater than 2. In this case, there is always a percolation transition. To see this, let us denote by $M(\mu)$ the left-hand side of Eq. (12):

$$M(\mu) = \sum_k \left\{ \frac{k(k-1)}{1 + k/A(\mu_c)} - k \right\} p_k. \tag{13}$$

If $M(\mu)>0$, the subnetwork percolates; if $M(\mu)<0$, it is fragmented. From Eq. (8), $A \to \infty$ as $\mu \to 0$, and thus

$$\lim_{\mu\to 0} M(\mu) = \sum_k \{k(k-1) - k\}p_k = \langle k^2\rangle - 2\langle k\rangle. \tag{14}$$

From the identity $\langle k^2\rangle \geq \langle k\rangle^2$, we conclude that for $\langle k\rangle > 2$, $\langle k^2\rangle > 2\langle k\rangle$. Hence, if the average degree is greater than 2, $\lim_{\mu\to 0} M(\mu)$ is always positive, and thus the free subnetwork percolates for sufficiently small particle density.

The other limit $\mu \to 1$, is similarly handled, using Eq. (8). We find that

$$\lim_{\mu\to 1} M(\mu) = -\sum_k k p_k = -\langle k\rangle < 0. \tag{15}$$

This means that for sufficiently high density, the free subnetwork is fragmented. Along with the result of the previous paragraph, this implies that $0<\mu_c<1$, as we announced. All important networks satisfy $\langle k\rangle > 2$, and therefore they all have a percolation transition in the free subnetwork. We note that this conclusion remains valid even if $\langle k^2\rangle$ diverges.

It is also interesting to consider the percolation properties of the occupied subnetwork, for which $\rho_k = n_k$ in Eq. (11). In this case, it is not true that there will always be a percolation transition. Take a general scale-free network, with a power-law degree distribution: $p_k \sim k^{-\gamma}$. One example is the Barabási-Albert network we use. For any fixed μ, A is a constant in the sum of Eq. (11). The term in the first sum behaves as $\sim k^{2-\gamma}$, and thus the sum diverges for $\gamma \leq 3$. This means that in this range of values of γ, the occupied subnetwork always percolates, most of the occupied nodes being concentrated on a single giant cluster with a finite fraction of the network. For this subnetwork there is no percolating transition, as it percolates for any $\mu>0$. This should be the case for the Barabási-Albert network we use in our examples, for which $\gamma=3$. This conclusion is confirmed by the results of the simulation: Fig. 3 shows that there is no range of μ for which M is zero.

VI. CONCLUSIONS

The transport model introduced and studied here is admittedly highly idealized: in real communication networks, information packets do not travel randomly, and the processing capacity of each node is neither homogeneous, nor limited to one packet. However, we think our model is worth studying as a minimal model for transport in networks in which "crowding" occurs, resulting from the interaction (in our

case, aversion) of the packets. Because of the simplicity of our model, we were able to obtain analytical results for important transport features. In particular, we predicted (and numerically confirmed) a percolation transition in the free subnetwork, and we are able to precisely quantify the way that inhomogeneities in the degree distribution (present in all real networks) affect traffic. These phenomena are clearly seen in our model as resulting from the inter-particle repulsion embodied in the Fermi-Dirac distribution, and this can be quantified precisely. In a purely numerical model, it would be hard to predict, for example, the percolating transition, and, more importantly, to understand precisely why it happens, as we can do with our model. Furthermore, we are confident that the collective phenomena of the model, especially the percolating transition, will be present in a more realistic model, albeit the details are certain to differ. Moreover, the present model may be generalized to include inhomogeneity in the processing capacity of the nodes, as well as other features. This line of research is being actively pursued.

ACKNOWLEDGMENTS

I would like to thank André Vieira for enlightening discussions. This work was partly supported by CNPq.

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